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The 3-D weight functions for a quasi-static planar crack

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Abstract

We explicitly evaluate the 3-D weight functions for a planar crack in an isotropic, homogeneous material; these give the full stress intensity factors induced by a static point force applied at an arbitrary position. If we Fourier decompose the 3-D weight functions with respect to the z variable then each Fourier mode satisfies the homogeneous equations of elasticity (except at the crack tip) and the boundary conditions on the crack face. Each Fourier mode diverges like $r^{-1/2}$ near the crack tip and decays exponentially for non-zero k_z. It is proved that these necessary conditions, which hold everywhere in the elastic material excluding the crack tip, are also sufficient to determine the 3-D weight functions. In particular, the 3-D weight functions can be calculated without considering an explicit loading problem.

Keywords: Fracture; Three dimensional weight function; Quasi-static planar crack

1. Introduction

In this publication, we calculate the quasi-static 3-D weight functions for a planar crack which extends to the left xz-plane ($x < 0$) with crack edge on the z-axis. The weight functions, or the Greens functions for a crack boundary problem, return the stress intensity factors (see Lawn (1993)) for a point loading. We will also need to distinguish three different cases. First, the weight functions for a planar crack in two dimensions which shall be denoted as 2-D weight functions. Second, the surface weight functions, that is the weight functions for forces applied only to the crack surfaces of a planar crack in three dimensions. Third, the fully 3-D weight functions giving the stress intensity distribution due to a (static) point force applied at an arbitrary position relative to a planar crack in three dimensions. We present the full 3-D weight function in explicit form, not least, because we shall need the explicit expression in a subsequent publication (Al-Falou and Ball, 2000) in which we investigate the path of a

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3-D crack near a point inhomogeneity. Furthermore, the 3-D weight for a planar crack provides the key to the perturbative expansion of weight functions for corrugated cracks. This relates very closely to the stability analysis of 3-D cracks. In the quasi-static case the stability of a slightly disturbed, initially planar 3-D crack can be read off the 3-D weight function. More details shall be given in a future paper on this subject. Additionally, the weight function method opens the prospect of a tractable stability analysis in the far more difficult case of a fast moving crack.

The existence of a surface weight function for a planar crack in two dimensions comes about by an application of Sochozki–Plemelj's formula (see the appendix on cracks in Mushkelishvili, 1963). If $g(x) = Q + iP$ is a balanced loading on the face of a planar crack (extending to the left $x < 0$) in two dimensions, then

$$K_{\mathrm{I}} - iK_{\mathrm{II}} = \frac{2}{\sqrt{2\pi i}} \int_{-\infty}^{0} \mathrm{d}x \frac{g(x)}{\sqrt{-x}}. \tag{1}$$

In this context, balanced loading means that the loading $Q + iP$ on the upper crack face $y = 0^{+}$ is balanced by an opposite loading $-(Q + iP)$ on the lower face $y = 0^{-}$. Bueckner (1970) has generalised the particular surface weight function in Eq. (1) to the full two-dimensional weight function (including other crack geometries). Bueckner and, later, Rice (1972) also pointed out that the full 2-D weight function satisfies the homogeneous equations of linear elasticity (everywhere except at the crack tip) and that the stress which it produces requires no body forces or boundary tractions for their equilibrium. We shall come back to this important feature of the 2-D and 3-D weight functions in the last section.

In a more recent publication, Bueckner also calculated the 3-D weight functions for a semi-infinite and a penny-shaped crack (Bueckner, 1987). An explicit form of the weight function is given for a point force acting on the crack surface. In our survey of previous work on weight functions we also refer the reader to Willis and Movchan who present a calculation for the 3-D dynamical weight functions which, of course, implies the quasi-static weight functions as a special case (Willis and Movchan, 1995). However, both authors state the weight functions only implicitly in terms of a convolution where each component of the convolution is given in integral form. In our computation of the 3-D weight functions we use an existing result on the surface weight function and a classical LEFM trick, that is, we can add a regular stress field without effecting the singular stress (if the boundary conditions are preserved). This leads to a significant simplification of our mathematical analysis in comparison with Bueckner's.

2. The surface weight functions

In our evaluation of the 3-D weight functions, we shall use an existing result on the surface weight function in three dimensions (Sih, 1973). The surface weight functions $\underline{W}$ return the stress intensity factors for a balanced point force (Q,P,R) at $(-x',0,z')$, that is, a pair of point forces where (Q,P,R) is applied at $(-x',0^{+},z')$ on the upper crack face and $(-Q,-P,-R)$ is applied at $(-x',0^{-},z')$ on the lower crack face (Fig. 1). Unfortunately, there are two sign errors in the formula for the weight functions in the handbook by Sih (1973), page 3.2.7–2. The following text should be read in conjunction with Fig. 2.

First, we consider the contribution to the mode II stress intensity from a balanced point force $(Q,0,R)$ at $(-a,0,0)$ (see above for the definition of balanced point force). If only a balanced point force $(Q,0,0)$ is applied [Fig. 2(a)] we obtain a positive contribution to K_{II}, which is in agreement with standard notations for the mode II stress intensity factor (Lawn, 1993). We now apply a balanced point force $(0,0,R)$ at $(-a,0,0)$ [Fig. 2(b)] which displaces the upper crack face in the positive z direction and the lower in the negative z direction, respectively. As a result of this shearing of the crack faces, we obtain a displacement on the upper crack face whose x-component is positive if z is positive and negative for

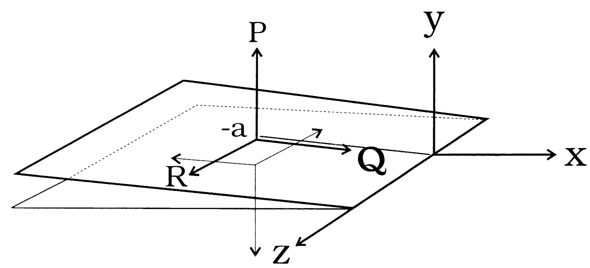

Fig. 1. Surface point forces for a planar crack in three dimensions.

negative z. This situation is reversed on the lower crack face. For positive z the displacement configuration is in agreement with the situation in Fig. 2(a) and, hence, the Q and R terms in the mode II weight function have the same sign for positive z. Note that the sign in Sih's handbook is wrong in this point.

Similarly, we obtain the proper signs for the mode III surface weight function. Thus, the corrected stress intensity factors (Sih, 1973) at $(0,0,Z)$ due to a pair of opposite point forces, i.e., (Q,P,R) applied at $(x',0^+,z')$ on the upper crack face and $(-Q,-P,-R)$ applied at $(x',0^-,z')$ on the lower crack face (x' is negative here), are given by

$$K_{\mathrm{I}}(x',z'-Z) = \underline{W}_{\mathrm{I}}(x',z'-Z)\cdot(Q,P,R) = P\Theta(-x')\frac{\sqrt{2\pi(-x')}}{\pi^2}\frac{1}{x'^2+(z'-Z)^2}, \tag{2}$$

$$\begin{aligned} K_{\mathrm{II}}(x',z'-Z) &= \underline{W}_{\mathrm{II}}(x',z'-Z)\cdot(Q,P,R) \\ &= Q\Theta(-x')\frac{\sqrt{2\pi(-x')}}{\pi^2}\frac{1}{x'^2+(z'-Z)^2}\left[1+\frac{2\nu}{2-\nu}\frac{x'^2-(z'-Z)^2}{x'^2+(z'-Z)^2}\right] \\ &\quad - R\Theta(-x')\frac{2(-x')\sqrt{2\pi(-x')}}{\pi^2}\frac{2\nu}{2-\nu}\frac{z'-Z}{(x'^2+(z'-Z)^2)^2} \end{aligned} \tag{3}$$

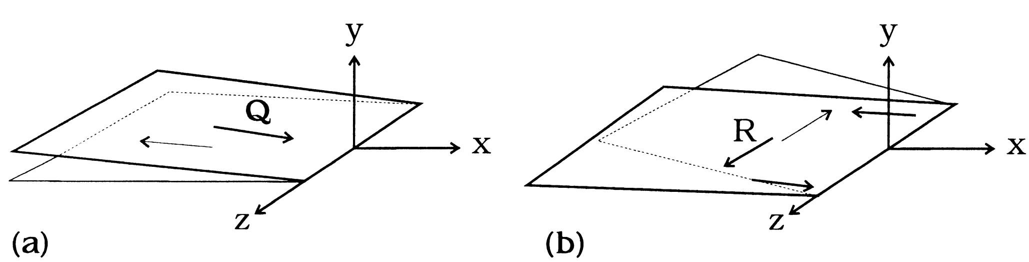

Fig. 2. Displacement fields for a balanced point force acting on the crack surface. The solid arrows show the displacements on the upper crack surface and the thin arrows on the lower crack face, respectively (note that the arrows do not show the components of the point force). In (a) the balanced point force points only in the x-direction, in (b) it points only in the z-direction (see also for the convention of the Q and R components of the point force).

and

$$K_{\mathrm{III}}(x',z'-Z) = \underline{W}_{\mathrm{III}}(x',z'-Z)\cdot(Q,P,R)$$
$$= R\Theta(-x')\frac{\sqrt{2\pi(-x')}}{\pi^2}\frac{1}{x'^2+(z'-Z)^2}\left[1-\frac{2\nu}{2-\nu}\frac{x'^2-(z'-Z)^2}{x'^2+(z'-Z)^2}\right]$$
$$-Q\Theta(-x')\frac{2(-x')\sqrt{2\pi(-x')}}{\pi^2}\frac{2\nu}{2-\nu}\frac{z'-Z}{(x'^2+(z'-Z)^2)^2}, \tag{4}$$

where $\Theta(x)$ is the theta function. Eqs. (2)–(4) define the surface weight functions $\underline{W}_{\mathrm{I}}(x',z'-Z)$, $\underline{W}_{\mathrm{II}}(x',z'-Z)$ and $\underline{W}_{\mathrm{III}}(x',z'-Z)$. It might be more intuitive to express the weight functions in terms of the variable $Z-z'$ but in the following we need to evaluate an integral over x',z'. This integral conveniently becomes a convolution integral if the x' and z' variables occur with the same sign. Later we shall need the Fourier transforms (with respect to $z'-Z$) of the surface weight functions in Eqs. (2)–(4). These can be obtained by contour integration in the complex plane and are given by

$$\underline{\hat{W}}_{\mathrm{I}}(k_x,k_z) = \sqrt{2|k_z|}(1-ik_x/|k_z|)^{-1/2}(0,1,0), \tag{5}$$

$$\underline{\hat{W}}_{\mathrm{II}}(k_x,k_z) = \sqrt{2|k_z|}\Bigg((1-ik_x/|k_z|)^{-1/2}+\frac{\nu}{2-\nu}(1-ik_x/|k_z|)^{-3/2},0,i\,\mathrm{sign}(k_z)\frac{\nu}{2-\nu}$$
$$\times(1-ik_x/|k_z|)^{-3/2}\Bigg) \tag{6}$$

and

$$\underline{\hat{W}}_{\mathrm{III}}(k_x,k_z) = \sqrt{2|k_z|}\Bigg(i\,\mathrm{sign}(k_z)\frac{\nu}{2-\nu}(1-ik_x/|k_z|)^{-3/2},0,(1-ik_x/|k_z|)^{-1/2}-\frac{\nu}{2-\nu}$$
$$\times(1-ik_x/|k_z|)^{-3/2}\Bigg) = i\,\mathrm{sign}(k_z)\left\{\frac{\sqrt{2|k_z|}}{(1-ik_x/|k_z|)^{1/2}}(1,0,i\,\mathrm{sign}(k_z))-\underline{\hat{W}}_{\mathrm{II}}(k_x,k_z)\right\}. \tag{7}$$

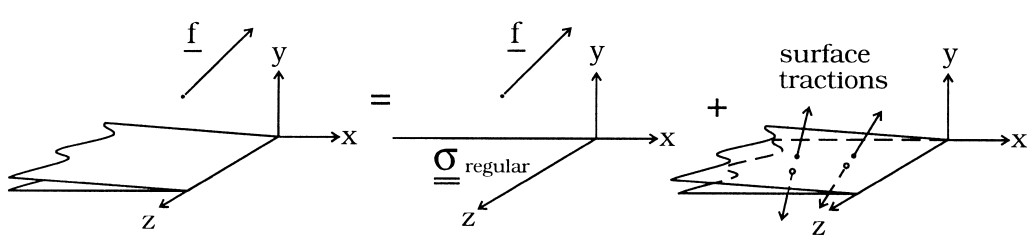

Fig. 3. Surface tractions due to a volume point force $\underline{f}$.

3. The Fourier transformed 3-D weight function for a planar crack

On the left hand side of Fig. 3, we see the basic problem in this publication. A quasi static planar crack extends to the left half plane in an infinite, isotropic and homogeneous linear elastic material. A volume force $\underline{f}$ is applied at the point (x_0,y_0,z_0). What are the induced singular stress fields? On the right hand side, the strategy to solve the problem is sketched. First, we evaluate the regular stress fields due to the point force as if there were no crack in the material. Second, we add the planar crack. The regular stress fields yield a balanced non-zero loading on the crack surface. In order to satisfy the boundary conditions on the crack faces, that is, zero tractions, we need to match this loading by a counter surface loading. We obtain the stress intensity factors generated by the counter tractions from the convolution integral of the surface loading and the surface weight function. This convolution integral is most conveniently evaluated in Fourier transformed coordinates. Note that have also added a table of symbols and notation in Appendix A.

All the steps in the following are given for the mode II weight function. The mode I and mode III weight functions are obtained in the same way (replace the index II by I or III). We are given a point volume force $\underline{f}\delta(\underline{x}'-\underline{x})$ at $\underline{x}$ in an infinite, isotropic, homogeneous elastic material. Our ansatz for the total stress field $\underline{\underline{\sigma}}_{\text{total}}$ is

$$\underline{\underline{\sigma}}_{\text{total}} = \underline{\underline{\sigma}}_{\text{regular}} + \underline{\underline{\sigma}}_{\text{s}}, \tag{8}$$

where $\underline{\underline{\sigma}}_{\text{regular}}$ is the (known) regular stress field for a point force in the absence of a crack and $\underline{\underline{\sigma}}_{\text{s}}$ is the remainder, whose singular part we want to calculate. The corresponding boundary conditions on the crack surface are

$$0 = \underline{\underline{\sigma}}_{\text{total}} \cdot \underline{n} = \underline{\underline{\sigma}}_{\text{regular}} \cdot \underline{n} + \underline{\underline{\sigma}}_{\text{s}} \cdot \underline{n}, \tag{9}$$

where $\underline{n}$ is the outer normal to the crack face. The last equation tells us that we must calculate the stress intensity factor generated by the tractions $-\underline{\underline{\sigma}}_{\text{regular}} \cdot \underline{n}$ on the crack surface. Note, only $\underline{\underline{\sigma}}_{\text{s}}$ contains the singular stress.

First, we need to state the regular stress in the absence of a crack. Let $\underline{\underline{G}}$ be the Green function for an infinite linear elastic material. By definition of this Green function, $\underline{\underline{G}}$, the displacement field $\underline{u}$ is given by

$$\underline{u}(\underline{x}') = \underline{\underline{G}}(\underline{x}' - \underline{x}) \cdot \underline{f} \tag{10}$$

and the corresponding stress tensor for the point volume force $\underline{f}\delta(\underline{x}'-\underline{x})$ is $\underline{\underline{\underline{\sigma}}}(\underline{x}'-\underline{x})\cdot\underline{f}$, where

$$\underline{\underline{\underline{\sigma}}}(\underline{x}' - \underline{x}) = \mathbf{C}:\nabla' \cdot \underline{\underline{G}}(\underline{x}' - \underline{x}). \tag{11}$$

The components of the tensor of elastic constants $\mathbf{C}$ are given by

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{12}$$

We now place the crack on the left half of the xz plane ($x < 0$, $y = 0$). The tractions of the regular stress field on the crack face are given by

$$\underline{t}(x',0,z',\underline{x}) = \left(\underline{\underline{\underline{\sigma}}}((x',0,z') - \underline{x}) \cdot \underline{f}\right) \cdot \underline{n}, \tag{13}$$

where $\underline{n}$ is the normal vector out of the crack surface.

This brings us to the second step in which the loading from the regular stress fields must be matched by a counter surface loading. According to Eq. (9), the stress intensity factors are obtained from convolution integrals of the negative loading in Eq. (13) and the surface weight functions. The stress intensity factor $K_{\rm II}$ for a point force $\underline{f}\delta(\underline{x}'-\underline{x})$ is given by the mode II surface weight function $\underline{W}_{\rm II}$,

$$K_{\rm II}(Z,x,y,z)=\int_S {\rm d}x'\,{\rm d}z'\,\underline{W}_{\rm II}(x',z'-Z)\cdot\left(-\underline{t}(x',z',\underline{x})\right), \tag{14}$$

where S denotes the fracture surface. The last equation, together with Eq. (13), implicitly gives the 3-D weight function because, for an arbitrary volume force $\underline{f}(x,y,z)$, the mode II stress intensity factor is obtained from

$$K_{\rm II}(Z)=\int_\Omega {\rm d}^3x\,\underline{\mathscr{G}}_{\rm II}(x,y,z-Z)\cdot\underline{f}(x,y,z). \tag{15}$$

Note that the regular stress fields automatically provide balanced tractions on the crack face. We finally obtain the 3-D weight function for mode II $\underline{\mathscr{G}}_{\rm II}$ by setting the (balanced) loading from Eq. (13) into Eq. (14),

$$\underline{\mathscr{G}}_{\rm II}(Z,x,y,z)=-\int_{-\infty}^{\infty}{\rm d}x'\int_{-\infty}^{\infty}{\rm d}z'\,\underline{W}_{\rm II}(x',z'-Z)\cdot\underline{\underline{\sigma}}(x'-x,-y,z'-z)\cdot\left(-\hat{e}_y\right), \tag{16}$$

where $-\hat{e}_y=(0,-1,0)$ is the outer normal to the upper crack face. Effectively, the x'-integral in Eq. (16) reduces to an integral from $-\infty$ to 0, since the surface weight functions are multiplied with $\Theta(-x')$. Noting that $\underline{\sigma}(x'-x,-y,z'-z)=-\underline{\sigma}(x-x',y,z-z')$ and changing the variable of z' integration, we can rewrite Eq. (16) as

$$\underline{\mathscr{G}}_{\rm II}(Z,x,y,z)=-\int_{-\infty}^{\infty}{\rm d}x'\int_{-\infty}^{\infty}{\rm d}z'\,\underline{W}_{\rm II}(x',z')\cdot\underline{\underline{\sigma}}(x-x',y,(z-Z)-z')\cdot\hat{e}_y. \tag{17}$$

Equivalent formulae hold for mode I and mode III. Eq. (17) is the central equation in this section. We note that the 3-D mode II weight function is a convolution integral, which suggests that it is more convenient to work with the Fourier transformed functions. This transforms (17) into

$$\hat{\underline{\mathscr{G}}}_{\rm II}(k_x,k_y,k_z)=-\hat{\underline{W}}_{\rm II}(k_x,k_z)\cdot\hat{\underline{\underline{\sigma}}}(k_x,k_y,k_z)\cdot\hat{e}_y, \tag{18}$$

where the Fourier transform in k_z is taken with respect to $z-Z$.

We shall first determine $\hat{\underline{\mathscr{G}}}(k_x,k_y,k_z)$ from $\hat{\underline{W}}_{\rm II}(k_x,k_z)$ and $\hat{\underline{\underline{\sigma}}}(k_x,k_y,k_z)$ and then perform the back Fourier transform. This rather technical calculation is given for the mode II weight function $\hat{\underline{\mathscr{G}}}_{\rm II}(k_x,k_y,k_z)$ in Appendix B in detail; the other two weight functions are obtained in a similar way and only their final form shall be given for the sake of completeness.

In the final result (Eq. (58) in Appendix B), we express the real space 3-D weight function $\underline{\mathscr{G}}_{\rm II}(x,y,k_z)$ (the z dependence remains Fourier transformed) in terms of a differential operator acting on a generating integral,

$$\begin{aligned}\mathscr{G}_{{\rm II},n}(x,y,k_z)=\sqrt{2|k_z|}\Bigg[&\frac{1}{2}\frac{1-2\nu}{1-\nu}\left(\frac{\nu}{2-\nu}-\partial_x\right)\delta_{2n}-\frac{y}{2}\frac{1}{1-\nu}\left(\frac{\nu}{2-\nu}-\partial_x\right)\partial_n-\partial_y\delta_{1n}\\&+\frac{2\nu}{2-\nu}\partial_\phi(\delta_{1n}+i\,{\rm sign}\,k_z\delta_{3n})\Bigg]\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}{\rm d}k_x\,{\rm d}k_y{\rm e}^{ik_xx}{\rm e}^{ik_yy}\frac{1}{(1+k^2)\sqrt{1-ik_x}}\end{aligned} \tag{19}$$

$n=1,2,3.$

Here ϕ is the angle in the x,y plane. Note that $\partial_3 = i\,\mathrm{sign}\,k_z$, and also that in Eqs. (19) and (20), the variables are dimensionless, as the old variables were replaced by dimensionless variables through division by $|k_z|$, that is, $k_x/|k_z| \to k_x$, $k_y/|k_z| \to k_y$, $k_z/|k_z| \to k_3$ and $x|k_z| \to x$, $y|k_z| \to y$ (see also Appendix B). It is shown in Appendix C that the generating integral in Eq. (19) is equal to

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \mathrm{d}k_x \mathrm{d}k_y \mathrm{e}^{ik_x x}\mathrm{e}^{ik_y y}\frac{1}{(1+k^2)\sqrt{1-ik_x}} = \frac{\mathrm{e}^{r\cos\phi}}{2\sqrt{2}}\mathrm{erfc}\left(\sqrt{2r}\cos\frac{\phi}{2}\right). \tag{20}$$

4. The real space mode II 3-D weight function for a planar crack

The derivatives in Eq. (19) were calculated with *Mathematica* (Wolfram Research, 1995). In the final result, we also reestablish the old variables $x \to x|k_z|$, etc., from the dimensionless variables. We finally obtain the 3-D weight function for mode II, where the z-dependence remains Fourier transformed:

$$\begin{aligned}\mathscr{G}_{\mathrm{II},x}(x,y,k_z) = \sqrt{|k_z|}\Bigg[&\big(8-12\nu+4\nu^2+(2-\nu)(\cos\phi+\cos 2\phi)+r|k_z|(-2+11\nu-8\nu^2 \\ &+2\nu\cos\phi+(2-\nu)\cos 2\phi)\big)\frac{\sin\dfrac{\phi}{2}}{4(1-\nu)(2-\nu)}\frac{\mathrm{e}^{-r|k_z|}}{\sqrt{2\pi r|k_z|}} \\ &+\frac{1-2\nu}{2-\nu}\frac{|k_z| r\sin\phi}{2}\mathrm{e}^{r|k_z|\cos\phi}\mathrm{erfc}\left(\sqrt{2r|k_z|}\cos\frac{\phi}{2}\right)\Bigg],\end{aligned} \tag{21}$$

$$\begin{aligned}G_{\mathrm{II},y}(x,y,k_z) = \sqrt{|k_z|}\Bigg[&\big(4-10\nu+4\nu^2+(2-\nu)(\cos\phi-\cos 2\phi)+r|k_z|(-2+3\nu+4(1-\nu) \\ &\times\cos\phi-(2-\nu)\cos 2\phi)\big)\frac{\cos\dfrac{\phi}{2}}{4(1-\nu)(2-\nu)}\frac{\mathrm{e}^{-r|k_z|}}{\sqrt{2\pi r|k_z|}}-\frac{1-2\nu}{2(2-\nu)}\mathrm{e}^{r|k_z|\cos\phi} \\ &\times\mathrm{erfc}\left(\sqrt{2r|k_z|}\cos\frac{\phi}{2}\right)\Bigg]\end{aligned} \tag{22}$$

and

$$\mathcal{G}_{\mathrm{II},z}(x,y,k_z) = i\,\mathrm{sign}(k_z)\sqrt{|k_z|}\left[\frac{\left(-1+\frac{5}{2}\nu-2\nu^2-\frac{2-\nu}{2}\cos\phi\right)\sin\frac{\phi}{2}}{(1-\nu)(2-\nu)}\sqrt{\frac{r|k_z|}{2\pi}}\mathrm{e}^{-r|k_z|}\right.$$
$$\left.+\frac{1-2\nu}{2-\nu}\frac{r|k_z|\sin\phi}{2}\mathrm{e}^{r|k_z|\cos\phi}\mathrm{erfc}\left(\sqrt{2r|k_z|}\cos\frac{\phi}{2}\right)\right], \tag{23}$$

where $x=r\cos\phi$ and $y=r\sin\phi$. The weight functions $\underline{\mathcal{G}}_{\mathrm{II},z}(x_0,y_0,k_z)$, as stated above, return the Fourier transformed mode II stress intensity factor $K_{\mathrm{II}}(k_z)$ for a volume force of the form $\underline{f}(\underline{x})=\underline{f}\delta(x-x_0)\delta(y-y_0)\mathrm{e}^{ik_z z}$. Fully in real space, $\underline{\mathcal{G}}_{\mathrm{II}}$ is given by

$$\underline{\mathcal{G}}_{\mathrm{II}}(Z,x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k_z\underline{\mathcal{G}}_{\mathrm{II}}(x,y,k_z)\mathrm{e}^{ik_z(z-Z)}, \tag{24}$$

where the variable k_z in $\underline{\mathcal{G}}_{\mathrm{II}}(x,y,k_z)$ indicates that here $\underline{\mathcal{G}}_{\mathrm{II}}$ is Fourier transformed in k_z. Furthermore, it is readily verified that for a balanced force on the crack faces, we recover the known surface weight function from the 3-D weight function in Eqs. (21)–(23). We have

$$\underline{\mathcal{G}}_{\mathrm{II}}(r,\phi=\pi,k_z)-\underline{\mathcal{G}}_{\mathrm{II}}(r,\phi=-\pi,k_z)=\underline{W}_{\mathrm{II}}(x,k_z), \tag{25}$$

where $\underline{W}_{\mathrm{II}}(x,k_z)$ is the surface weight function for a balanced point force.

5. The mode I and mode III 3-D weight functions for a planar crack

The 3-D weight functions for mode I and mode III are obtained in a similar way. The result for mode I is

$$\mathcal{G}_{\mathrm{I},x}(x,y,k_z) = \sqrt{|k_z|}\left[\frac{1-2\nu}{4(1-\nu)}\mathrm{e}^{r|k_z|\cos\phi}\mathrm{erfc}\left(\sqrt{2r|k_z|}\cos\frac{\phi}{2}\right)+\mathrm{e}^{-r|k_z|}\frac{\cos\left(\frac{\phi}{2}\right)}{4(1-\nu)\sqrt{2\pi r|k_z|}}(-2+4\nu\right.$$
$$\left.+\cos\phi-\cos(2\phi)+r|k_z|(-1+2\cos\phi-\cos(2\phi)))\right], \tag{26}$$

$$\mathcal{G}_{\mathrm{I},y}(x,y,k_z) = \sqrt{|k_z|}\mathrm{e}^{-r|k_z|}\frac{\sin\frac{\phi}{2}}{4(1-\nu)\sqrt{2\pi r|k_z|}}(4-4\nu-\cos\phi-\cos(2\phi)+r|k_z|(1-\cos(2\phi))) \tag{27}$$

and

$$\mathcal{G}_{\mathrm{I},z}(x,y,k_z) = i\,\mathrm{sign}(k_z)\sqrt{|k_z|}\left[\frac{-\sin\frac{\phi}{2}\sin\phi}{2(1-\nu)}\sqrt{\frac{r|k_z|}{2\pi}}\mathrm{e}^{-r|k_z|}+\frac{(1-2\nu)}{4(1-\nu)}\mathrm{e}^{r|k_z|\cos\phi}\mathrm{erfc}\left(\sqrt{2r|k_z|}\cos\frac{\phi}{2}\right)\right], \tag{28}$$

and, for mode III, we obtain

$$\mathscr{G}_{\mathrm{III},x}(x,y,k_z) = -i\,\mathrm{sign}(k_z)\sqrt{|k_z|}\left[\frac{\sqrt{r|k_z|}(1-2\nu+\cos(\phi))\sin\left(\frac{\phi}{2}\right)}{(2-\nu)\sqrt{2\pi}}\mathrm{e}^{-r|k_z|} - \frac{\mathrm{e}^{r|k_z|\cos(\phi)}(1-2\nu)r|k_z|\mathrm{erfc}\left(\sqrt{2r|k_z|}\cos\left(\frac{\phi}{2}\right)\right)\sin(\phi)}{2(2-\nu)}\right], \tag{29}$$

$$\mathscr{G}_{\mathrm{III},y}(x,y,k_z) = -i\,\mathrm{sign}(k_z)\sqrt{|k_z|}\left[\frac{\sqrt{\frac{2}{\pi}}\sqrt{r|k_z|}\cos\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right)^2}{(2-\nu)}\mathrm{e}^{-r|k_z|} + \frac{\mathrm{e}^{r|k_z|\cos(\phi)}(1-2\nu)\mathrm{erfc}\left(\sqrt{2r|k_z|}\cos\left(\frac{\phi}{2}\right)\right)}{2(2-\nu)}\right] \tag{30}$$

and

$$\mathscr{G}_{\mathrm{III},z}(x,y,k_z) = \sqrt{|k_z|}\left[\frac{(2-\nu-2\nu r|k_z|)\sin\left(\frac{\phi}{2}\right)}{(2-\nu)\sqrt{2\pi}\sqrt{r|k_z|}}\mathrm{e}^{-r|k_z|} - \frac{\mathrm{e}^{r|k_z|\cos(\phi)}(1-2\nu)r|k_z|\mathrm{erfc}\left(\sqrt{2r|k_z|}\cos\left(\frac{\phi}{2}\right)\right)\sin(\phi)}{2(2-\nu)}\right]. \tag{31}$$

6. Properties and uniqueness of the 3-D weight function

We now shall further explore the properties of the 3-D weight function for a planar crack. We shall see that we can uniquely determine the 3-D weight function without considering an explicit loading problem. This opens another approach to the evaluation of weight functions (from our calculation) which proves particularly valuable in the more difficult dynamical case (this work is in progress at the moment). It also provides a check on the 3-D weight functions themselves. Furthermore, we shall link the results in this thesis to a former calculation of Ball and Larralde (1995). We shall see in the following that the mode I and mode II weight functions are implicitly contained in their calculation.

First, we investigate the necessary properties of the weight function. We then shall see that these properties are also sufficient to establish uniqueness of the weight function, and hence these properties determine the weight function uniquely.

In the following, we suppose that there exist several 3-D weight functions for a planar crack J and we ask about their common properties. First, we observe that the 3-D weight function satisfies the homogeneous equations of elasticity and the boundary conditions on the crack surface. Strictly, the 3-D weight function has to satisfy the adjoint equations of linear homogeneous elasticity. However, the linear elasticity operator is self-adjoint (see Appendix D). We then have to solve the displacement field in the presence of a volume force $\underline{f}$ at $\underline{x}_0$,

$$-\mu\left(\Delta+\frac{1}{1-2\nu}\nabla\,\mathrm{div}\right)\underline{u}=\delta(\underline{x}-\underline{x}_0)\underline{f}\text{ or }E\underline{u}=\delta(\underline{x}-\underline{x}_0)\underline{f}, \tag{32}$$

where E is the operator of linear elasticity for an isotropic, homogeneous material, given in the left equation. The displacement $\underline{u}$ must also satisfy the boundary conditions on the crack surface, i.e.,

$$\underline{\underline{\sigma}}[\underline{u}]\cdot\underline{n}=0\text{ on the crack surface.} \tag{33}$$

Indeed, in this way, Kassir and Sih calculated their surface weight function.

Although Eq. (32) might suggest that u is a function of $\underline{x}-\underline{x}_0$, this is not the case because the boundary conditions in Eq. (33) are not translationally invariant. As we shall see in the following, we nevertheless have

$$\underline{u}(\underline{x},\underline{x}_0)=\underline{u}(\underline{x}_0,\underline{x}), \tag{34}$$

which we now show follows from the self-adjointness of the operator E of linear elasticity (see Appendix D for a proof of the self-adjointness of E). Let Ω be $\mathbb{R}^3$ excluding the left xz-plane (i.e., the crack surface), that is, the domain occupied by the elastic material. We define a scalar product for functions on Ω by

$$\langle\underline{f},\underline{g}\rangle_2=\int_\Omega \mathrm{d}^3x\,\underline{f}(\underline{x})\cdot\underline{g}(\underline{x}). \tag{35}$$

Furthermore, let $\underline{u}(\underline{x},\underline{x}_0)$ and $\underline{u}(\underline{x},\underline{x}_1)$ be two solutions of Eq. (33) with $\underline{x}_0$ replaced by $\underline{x}_1$ in the latter. By definition of the self-adjointness of the operator E, we have

$$\langle\underline{u}(\underline{x},\underline{x}_0),E\underline{u}(\underline{x},\underline{x}_1)\rangle_2=\langle E\underline{u}(\underline{x},\underline{x}_0),\underline{u}(\underline{x},\underline{x}_1)\rangle_2, \tag{36}$$

hence,

$$\langle\underline{u}(\underline{x},\underline{x}_0),\delta(\underline{x}-\underline{x}_1)\underline{f}\rangle_2=\langle\delta(\underline{x}-\underline{x}_0)\underline{f},\underline{u}(\underline{x},\underline{x}_1)\rangle_2, \tag{37}$$

hence,

$$\underline{u}(\underline{x}_1,\underline{x}_0)\cdot\underline{f}=\underline{u}(\underline{x}_0,\underline{x}_1)\cdot\underline{f}\text{ for all }\underline{f}, \tag{38}$$

which proves Eq. (34).

Assuming the displacement $\underline{u}$ and the corresponding stress field $\underline{\underline{\sigma}}(\underline{x},\underline{x}')$ have been obtained, we can extract the 3-D weight functions by taking appropriate limits in $\underline{x}$, e.g.,

$$K_{\mathrm{II}}(\underline{x}_0,\underline{f})=\lim_{x\to0^+}\sqrt{2\pi x}\sigma_{r,\phi}(\underline{x},\underline{x}_0,\underline{f})|_{y=0}=\lim_{x\to0^+}\sqrt{2\pi x}\sigma_{r,\phi}(\underline{x}_0,\underline{x},\underline{f})|_{y=0}. \tag{39}$$

The last equation holds since $\underline{u}(\underline{x},\underline{x}_0)=\underline{u}(\underline{x}_0,\underline{x})$. We conclude that $K_{\mathrm{II}}(\underline{x}_0,\underline{f})$, and thus each component of the 3-D weight function, satisfy the equations of elasticity in Eq. (32) and the boundary conditions, where derivatives are taken with respect to $\underline{x}_0$.

Second, we exploit the fact that the problem of a planar crack in an infinite, isotropic, homogeneous material has only two variables with the dimension of length. A 3-D weight function $\underline{J}(z-Z,r,\phi)$ has dimension length $\ell^{-3/2}$. If we Fourier transform it with respect to $z-Z$, each Fourier component $\underline{J}(k_z,r,\phi)$ has dimension length $\ell^{-1/2}$ (exceptionally, we denote the real space function and the Fourier transformed function by the same symbol; distinction is made by the dependence on k_z). However, the only quantities with dimension length are the radial coordinate r and the inverse of the wavenumber k_z. Considering the dimension of $\underline{J}(k_z,r,\phi)$, the only possible form it can have is

$$\underline{J}(k_z,r,\phi) = \sqrt{|k_z|}\underline{J}_1(r \cdot k_z,\phi). \tag{40}$$

It is reasonable to assume that the weight function exists for $k_z = 0$ and is non-zero. Indeed, this is the two-dimensional weight function which can be obtained in a straightforward calculation. We then deduce two results from Eq. (40). First, the highest order divergence near the crack tip is $1/\sqrt{r}$, and second, the $1/\sqrt{r}$ term does not depend on k_z. In other words, the $1/\sqrt{r}$ divergence in $\underline{J}(k_z,r,\phi)$ is the same for all k_z and equals $\underline{J}_0(r,\phi) = \underline{J}(0,r,\phi)$. In summary, we have the following properties of 3-D weight functions for a planar crack in an infinite linear elastic medium

1. $\underline{\mathscr{G}}_{k_z}$ satisfies the homogeneous equations of elasticity and the boundary conditions on the crack face for all k_z.
2. $\underline{\mathscr{G}}_{k_z}$ diverges like $r^{-1/2}$ near $r = 0$ independently of k_z.
3. The divergent part of $\underline{\mathscr{G}}_{k_z}$ equals the (k_z-independent) 2-D weight function for all k_z.

So far we have had to assume that there might exist several weight functions. We now want to show uniqueness. In the following, we only admit weight functions $\underline{J}(k_z,r,\phi)$ that diverge like $1/\sqrt{r}$ near $r = 0$ and fall off exponentially as $r \to \infty$ for $k_z \neq 0$ (indeed, we do not need exponential decay, as we shall see). Furthermore, we assume that two possibly different Fourier transformed 3-D weight functions $\underline{J}_1(k_z,r,\phi)$ and $\underline{J}_2(k_z,r,\phi)$ are equal for $k_z = 0$, i.e. $\underline{J}_1(0,r,\phi) = \underline{J}_2(0,r,\phi)$, which implies their divergent parts are the same. Then we define

$$\underline{F}(\underline{x}) = \left(\underline{J}_1(k_z,r,\phi) - \underline{J}_2(k_z,r,\phi)\right)\mathrm{e}^{ik_z z}. \tag{41}$$

Note that $\underline{F}(\underline{x})$ no longer diverges near $r = 0$ and, more importantly, it has finite elastic energy. Note also that $\underline{F}(\underline{x})$ satisfies the equations of linear elasticity and the boundary conditions on the crack face since $\underline{J}_1(k_z,r,\phi)$ and $\underline{J}_2(k_z,r,\phi)$ have these properties. To make further progress, we use the elastic energy in order to apply a norm argument. We define

$$\langle \underline{u},\underline{u}\rangle_E := \int_\Omega \mathrm{d}^3x(\nabla\underline{u}):\mathbf{C}:(\nabla\underline{u}) = \int_\Omega \mathrm{d}^3x(\nabla\underline{u}):\underline{\underline{\sigma}}[\underline{u}], \tag{42}$$

where $\mathbf{C}$ is the tensor of elastic constants (see Eq. (11)), $\underline{\underline{\sigma}}[\underline{u}]$ is the stress tensor generated by u and Ω is the region occupied by the elastic material. Inserting F in Eq. (42), we obtain

$$\langle \underline{F},\underline{F}\rangle_E = \int_{\Omega_R} \mathrm{d}^3x[\nabla\cdot(\underline{F}\cdot\underline{\underline{\sigma}}[\underline{F}]) - \underline{F}\cdot(\nabla\cdot\underline{\underline{\sigma}}[\underline{F}])], \tag{43}$$

$$\langle \underline{F},\underline{F}\rangle_E = \int_{\partial\Omega_R} \mathrm{d}\underline{S}\cdot\underline{\underline{\sigma}}[\underline{F}]\cdot\underline{F} \tag{44}$$

and

$$\langle \underline{F},\underline{F}\rangle_E = \int_{\Gamma_R} \mathrm{d}\underline{S}\cdot\underline{\underline{\sigma}}[\underline{F}]\cdot\underline{F} \to 0 \text{ as } R\to\infty, \tag{45}$$

where Γ_R is the circle with radius R. The contour $\partial\Omega_R$ is shown in Fig. 4. Eq. (44) holds since F satisfies the homogeneous equations of elasticity. In Eq. (45), we have used the fact that F satisfies the boundary conditions on the crack surface. The limit is zero since we have assumed that both weight functions decay exponentially for $k \neq 0$ as $r \to \infty$. As mentioned above, it would be sufficient to require decay like $r^{-\alpha}$ where $\alpha > 1/2$. In any case, we have that $\langle F,F\rangle_\mathrm{E} = 0$.

We now want to bound the elastic energy $\langle u,u\rangle_\mathrm{E}$. The elastic energy can be rewritten in terms of the strain tensor $\underline{e} = 1/2(\nabla\underline{u} + (\nabla\underline{u})^T)$ as

$$\langle \underline{u},\underline{u} \rangle_E = \int_\Omega \mathrm{d}^3x \{ \lambda \, \mathrm{tr}(\underline{e}[\underline{u}])\mathrm{tr}(\underline{e}[\underline{u}]) + 2\mu \underline{e}[\underline{u}]:\underline{e}[\underline{u}] \}, \tag{46}$$

which is non-negative and zero only if $\underline{e}[\underline{u}]$ is zero almost everywhere. Since $\langle \underline{F},\underline{F} \rangle_E = 0$, we deduce that $\underline{e}[F] = 0$ almost everywhere. It follows (for example, see page 293 in Ciarlet, 1993) that

$$\underline{F} = \underline{a} + \underline{b} \wedge \underline{x}. \tag{47}$$

Remembering that $\underline{F}(\underline{x})$ decays to zero as $\underline{x} \to \infty$, we conclude that $\underline{F}$ is identically zero. Hence, $\underline{J}_1(k_z,r,\phi) = \underline{J}_2(k_z,r,\phi)$ provided that the $r^{-1/2}$ terms of weight function 1 and 2 are equal.

We can summarise our results as follows. The 3-D weight functions for a planar crack in an isotropic, homogeneous material are uniquely determined (up to a constant pre factor) by the following requirements on its Fourier components $\underline{\mathscr{G}}_{k_z}$ with respect to z:

1. $\underline{\mathscr{G}}_{k_z}$ satisfies the homogeneous equations of elasticity and the boundary conditions on the crack face for all k_z.
2. $\underline{\mathscr{G}}_{k_z}$ decays exponentially for $r \to \infty$ for $k_z \neq 0$.
3. $\underline{\mathscr{G}}_{k_z}$ diverges like $r^{-1/2}$ near $r = 0$ independently of k_z.
4. The divergent part of $\underline{\mathscr{G}}_{k_z}$, i.e., $\underline{\mathscr{G}}_0$, is uniquely given.

We have now reduced the uniqueness of the 3-D weight functions to the uniqueness of the 2-D weight functions. Let us assume that we have evaluated functions $\underline{\mathscr{G}}_{k_z}$ which satisfy conditions 1 to 3. To pick out the mode II 3-D weight function $\underline{\mathscr{G}}_{\mathrm{II},k_z}$, for example, we compare the divergent part with the known 2-D mode II weight function in the x,y plane ($k_z = 0$). Note that no point force occurs explicitly anywhere in the conditions 1 to 3. Note also that we mentioned in the beginning of this chapter that the 3-D weight functions satisfy the first condition above. In other words, the 3-D weight function can be calculated uniquely on the basis of conditions 1 to 3 without taking into consideration any forces or tractions. This fact opens different approach in the calculation of the 3-D weight function from the one presented in this paper.

What happens if we work purely on the basis of conditions 1 to 3, i.e., the 2-D weight functions in the x,y plane are not given? Then $\underline{\mathscr{G}}_{k_z}$ can be uniquely determined even if we work only with the homogeneous equations of linear elasticity (without considering an explicit loading problem).

The $r^{-1/2}$ term equals the Fourier mode $\underline{G}_0(r,\phi)$ for $k_z = 0$, which of course satisfies the equation of

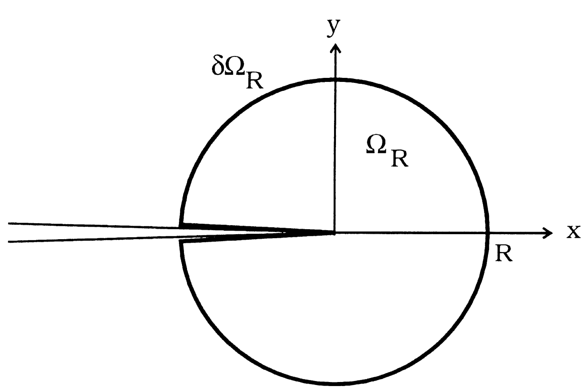

Fig. 4. Region and contour of integration ($R \to \infty$).

elasticity and the boundary conditions. However, there is no z dependence and we know the r dependence of $\underline{G}_0(r,\phi)$, i.e., $\underline{G}_0(r,\phi)=\underline{j}(\phi)/\sqrt{r}$. We also know (Freund, 1990) that a solution $\underline{G}_0(r,\phi)$ of the two-dimensional linear elastic equations with zero forces on the crack surface and $r^{-1/2}$ dependence is a linear combination of the form $\alpha_{\mathrm{I}}/\sqrt{r}\underline{j}_{\mathrm{I}}(\phi)+\alpha_{\mathrm{II}}/\sqrt{r}\underline{j}_{\mathrm{II}}(\phi)+\alpha_{\mathrm{III}}/\sqrt{r}\underline{j}_{\mathrm{III}}(\phi)$, where $\underline{j}_{\mathrm{I}}$, $\underline{j}_{\mathrm{II}}$ and $\underline{j}_{\mathrm{III}}$ are the angular parts of the 2-D weight functions.

On the other hand, any function $\underline{\mathscr{G}}_{k_z}$ which meets the requirements 1 to 3 is uniquely determined by the divergent part $\underline{\mathscr{G}}_0$. With the previous statement, we conclude that any such function is a linear combination of the form

$$\underline{\mathscr{G}}_{k_z} = \alpha_{\mathrm{I}}\underline{\mathscr{G}}_{\mathrm{I},k_z} + \alpha_{\mathrm{II}}\underline{\mathscr{G}}_{\mathrm{II},k_z} + \alpha_{\mathrm{III}}\underline{\mathscr{G}}_{\mathrm{III},k_z}, \tag{48}$$

where α_{I}, α_{II} and α_{III} are real numbers and $\underline{\mathscr{G}}_{\mathrm{I},k_z}$, $\underline{\mathscr{G}}_{\mathrm{II},k_z}$ and $\underline{\mathscr{G}}_{\mathrm{III},k_z}$ are the 3-D weight functions for modes I to III. We can extract a single mode 3-D weight function (up to constant pre factors) out of Eq. (48) on the basis of symmetry arguments. For example, the first component of $\underline{\mathscr{G}}_{\mathrm{I},k_z}$ is even in y and z, the second component is odd in y and even in z and the third component is even in y and odd in z. The remaining real pre factor can be fixed by applying the weight function to a loading with known stress intensity factors or by comparison with the 2-D weight function.

The preceding discussion links the calculation of the 3-D weight functions to a result obtained by Ball and Larralde (1995), who investigated the stability of slow cracks under mode I loading. For this purpose, they expanded the perturbed stress field in terms of the unperturbed stress field plus a first order contribution $\underline{\underline{\sigma}}_1(\underline{x})$ which had to be determined, i.e.,

$$\underline{\underline{\sigma}}\left(\underline{x} + \hat{e}_z h_{k_z}(x)\mathrm{e}^{ik_z z}\right) = \underline{\underline{\sigma}}_0(\underline{x}) + h_{k_z}(x)\mathrm{e}^{ik_z z}(\hat{e}_z \cdot \nabla)\underline{\underline{\sigma}}_0(\underline{x}) + \underline{\underline{\sigma}}_1(\underline{x}), \tag{49}$$

where $\hat{e}_z h_{k_z}(x)\mathrm{e}^{ik_z z}$ denotes the (small) deviation of the crack surface from planarity. It is clear that $\underline{\underline{\sigma}}_1(\underline{x})$ diverges like $r^{-3/2}$ and the associated displacement field $\underline{u}_1(\underline{x})$ like $r^{-1/2}$. Their first order displacement field satisfies the equations of elasticity and the boundary conditions. Furthermore, the Fourier components fall off exponentially for $k_z \neq 0$. Hence, the 3-D weight functions are implicitly contained in the first order displacement field. The single components for mode I and mode II can be extracted by symmetry arguments and yield the same 3-D weight functions as given here by Eqs. (26)–(28) and Eqs. (21)–(23).

7. Conclusions

We have calculated the full 3-D weight functions for mode I, mode II and mode III for a quasi-static planar crack using an existing result on the surface weight functions. From another point of view, we have shown that weight functions are governed by uniqueness requirements: First, $\underline{\mathscr{G}}_{k_z}$ has to satisfy the homogeneous equations of elasticity and the boundary conditions on the crack face for all k_z. Second, $\underline{\mathscr{G}}_{k_z}$ needs to decay exponentially for $r \to \infty$ for $k_z \neq 0$. Finally, $\underline{\mathscr{G}}_{k_z}$ has to diverge like $1/\sqrt{r}$ near $r = 0$ independently of k_z, and this divergent part is uniquely given (by the 2-D weight function). The first condition is long established (Bueckner, 1970; Rice, 1972); the second is required for any physically sensible solution; the third we have demonstrated. Hence, we can determine a weight function by solving the homogeneous linear elastic equations of elasticity with zero boundary tractions and the additional requirement that the highest order divergence is $1/\sqrt{r}$ near the crack tip. Using this method, it is possible to circumvent the evaluation of a more difficult explicit loading problem.

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Appendix A. Nomenclature

$\mathbf{C}$	tensor of elastic constants, e.g., for an isotropic material with components $C_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$
E	operator of linear elasticity for an isotropic, homogeneous material, $E=-\nabla\cdot\mathbf{C}:\nabla$
erf(x), erfc(x)	error function and complementary error function
$\underline{G}$	Greens function for the displacement field in an infinite linear elastic isotropic and homogeneous material
$\hat{e}_x$, $\hat{e}_y$, $\hat{e}_z$	Cartesian vectors, $\hat{e}_x=(1,0,0)$, $\hat{e}_y=(0,1,0)$, $\hat{e}_z=(0,0,1)$
$\underline{\underline{e}}[\underline{u}]$	strain tensor generated by $\underline{u}$
$\hat{f}$	Fourier transform of the function f
$\underline{\mathscr{G}}_{\mathrm{I}}$, $\underline{\mathscr{G}}_{\mathrm{II}}$, $\underline{\mathscr{G}}_{\mathrm{III}}$	three-dimensional weight function for mode I, mode II and mode III
$\underline{n}$	normal vector
ν	Poisson's ratio
K_{I}, K_{II}, K_{III}	stress intensity factors for mode I, mode II and mode III
λ, μ	Lame coefficients
Ω	domain occupied by the elastic material
$\partial\Omega$	boundary of the domain Ω
ϕ	angle in the x,y plane, $x=r\cos\phi$, $y=r\sin\phi$
$\underline{\underline{\sigma}}(\underline{x})$	stress tensor at $\underline{x}$
$\underline{W}_{\mathrm{I}}$, $\underline{W}_{\mathrm{II}}$, $\underline{W}_{\mathrm{III}}$	surface weight function for mode I, mode II and mode III in three dimensions
$0^{\pm}$	indicates the positive or negative limit to zero, usually used in the context of the upper ($y=0^+$) or lower crack face ($y=0^-$)
$\times$	multiplication
$\wedge$	curl
$\langle .,. \rangle_2$	L^2 scalar product defined in Eq. (35)
$\langle .,. \rangle_{\mathrm{E}}$	scalar product defined in Eq. (42)

The Fourier transform and the back Fourier transform are defined as

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k\mathrm{e}^{ikx}\hat{f}(k)$$

and

$$\hat{f}(k)=\int_{-\infty}^{\infty}\mathrm{d}x\mathrm{e}^{-ikx}\hat{f}(x) \tag{50}$$

throughout this thesis. Occasionally, the Fourier transform is indicated by the argument instead of the hat.

Appendix B. Evaluation of the Fourier transformed 3-D weight function and the back Fourier transformation

We start from Eq. (18), stating the Fourier transformed mode II 3-D weight function in terms of the known Fourier transformed surface weight function and the Fourier transformed stress field for a point force,

$$\underline{\hat{\mathscr{G}}}_{\mathrm{II}}(k_x,k_y,k_z) = -\underline{\hat{W}}_{\mathrm{II}}(k_x,k_z)\cdot \underline{\underline{\underline{\hat{\sigma}}}}(k_x,k_y,k_z)\cdot \hat{e}_y. \tag{51}$$

The Fourier transform of the Green function of an infinite isotropic homogeneous linear elastic material is

$$\hat{G}_{ij} = \frac{1}{\mu}\frac{1}{k_z^2+k^2}\left(\delta_{ij} - \frac{1}{2(1-\nu)}\frac{k_ik_j}{k_z^2+k^2}\right)\ (i,j = x,y,z \text{ or } 1,2,3), \tag{52}$$

where $k^2 = k_x^2 + k_y^2$. From Eq. (11), namely $\sigma_{ijn} = C_{ijkl}\partial_k G_{ln}$, we obtain

$$\hat{\sigma}_{ijn} = i\frac{1}{k_z^2+k^2}\left(\frac{\nu}{1-\nu}k_n\delta_{ij} + k_i\delta_{jn} + k_j\delta_{in} - \frac{1}{1-\nu}\frac{k_ik_jk_n}{k_z^2+k^2}\right). \tag{53}$$

In particular, the Fourier transform $\underline{\underline{\underline{\hat{\sigma}}}}(k_x,k_y,k_z)\cdot \hat{e}_y$ is

$$\hat{\sigma}_{i2n} = i\frac{1}{k_z^2+k^2}\left(\frac{\nu}{1-\nu}k_n\delta_{i2} + k_i\delta_{2n} + k_y\delta_{in} - \frac{1}{1-\nu}\frac{k_ik_yk_n}{k_z^2+k^2}\right). \tag{54}$$

The Fourier transformed mode II surface weight function is given in Eq. (6)

$$\begin{aligned}\underline{\hat{W}}_{\mathrm{II}}(k_x,k_z) = \sqrt{2|k_z|}\Bigg(&(1-ik_x/|k_z|)^{-1/2} + \frac{\nu}{2-\nu}(1-ik_x/|k_z|)^{-3/2}, 0,\\ &i\,\mathrm{sign}(k_z)\frac{\nu}{2-\nu}(1-ik_x/|k_z|)^{-3/2}\Bigg).\end{aligned} \tag{55}$$

Eqs. (54) and (55) are set into Eq. (51) in order to yield the Fourier transformed mode II 3-D weight function,

$$\begin{aligned}\hat{\mathscr{G}}_{\mathrm{II},n}(k_x,k_y,k_z) = \frac{-i\sqrt{2|k_z|}}{k_z^2+k^2}\Bigg\{&\left(k_x\delta_{2n} + k_y\delta_{1n} - \frac{1}{1-\nu}\frac{k_xk_yk_n}{k_z^2+k^2}\right)\\ &\times\left(\frac{1}{\sqrt{1-i\frac{k_x}{|k_z|}}} + \frac{\frac{\nu}{2-\nu}}{\left(1-i\frac{k_x}{|k_z|}\right)^{3/2}}\right) + \left(k_z\delta_{2n} + k_y\delta_{3n} - \frac{1}{1-\nu}\frac{k_zk_yk_n}{k_z^2+k^2}\right)\\ &\times i\,\mathrm{sign}(k_z)\frac{\frac{\nu}{2-\nu}}{\left(1-i\frac{k_x}{|k_z|}\right)^{3/2}}\Bigg\}.\end{aligned} \tag{56}$$

The remaining work consists of back Fourier transforming this expression in order to obtain $\underline{\mathscr{G}}_{\rm II}(x,y,k_z)$ (we drop the hat here although the 3-D weight function remains Fourier transformed in k_z).

The main steps of the back Fourier transformation in k_x and k_y will be shown for the mode II weight function $\hat{\underline{\mathscr{G}}}_{\rm II}(k_x,k_y,k_z)$. The other two weight functions are obtained in a similar way and only their final form shall be given for the sake of completeness.

First, we observe that it is convenient to replace the old variables by dimensionless variables through division by $|k_z|$, that is, $k_x/|k_z| \to k_x$, $k_y/|k_z| \to k_y$, $k_z/|k_z| \to k_3$ and $x|k_z| \to x$, $y|k_z| \to y$. Second, the terms $(1-ik_x/|k_z|)^{-3/2}$ are reduced to $(1-ik_x/|k_z|)^{-1/2}$ by multiplication with $1-ik_x/|k_z|$. Furthermore, we observe that integration by parts with respect to k_y turns the terms $\frac{k_y k_n}{(1+k^2)^2}$ into $i\frac{y}{2}\frac{k_n}{1+k^2} + \frac{1}{2}\frac{\sigma_{2n}}{1+k^2}$. Thus, we obtain

$$\mathscr{G}_{{\rm II},n}(x,y,k_z) = \frac{-i\sqrt{2|k_z|}}{(2\pi)^2}\int_{-\infty}^{\infty} {\rm d}k_x\,{\rm d}k_y {\rm e}^{ik_x x}{\rm e}^{ik_y y}\frac{1}{1+k^2}\left\{\left[\frac{1}{2}\frac{1-2\nu}{1-\nu}\left(k_x + i\frac{\nu}{2-\nu}\right)\delta_{2n} - \frac{iy}{2}\frac{1}{1-\nu}\right.\right.$$
$$\left.\left.\times\left(k_x + i\frac{\nu}{2-\nu}\right)k_n + k_y\delta_{1n}\right]\frac{1}{\sqrt{1-ik_x}} + \frac{\nu}{2-\nu}k_y\left[\delta_{1n} + i\,{\rm sign}\,k_z\delta_{3n}\right]\frac{1}{(1-ik_x)^{3/2}}\right\}. \tag{57}$$

Integration by parts first with respect to k_x and then with respect to k_y changes the term $\frac{k_y}{(1+k^2)(1-ik_x)^{3/2}}$ into $-2\frac{xk_y-yk_x}{(1+k^2)(1-ik_x)^{1/2}} = 2i\frac{\partial_\phi}{(1+k^2)(1-ik_x)^{1/2}}$. Here ϕ is the angle in the x,y plane, i.e., $x=\sqrt{x^2+y^2}\cos\phi$ and $y=\sqrt{x^2+y^2}\sin\phi$. We obtain

$$\mathscr{G}_{{\rm II},n}(x,y,k_z) = \sqrt{2|k_z|}\left[\frac{1}{2}\frac{1-2\nu}{1-\nu}\left(\frac{\nu}{2-\nu} - \partial_x\right)\delta_{2n} - \frac{y}{2}\frac{1}{1-\nu}\left(\frac{\nu}{2-\nu} - \partial_x\right)\partial_n - \partial_y\delta_{1n}\right.$$
$$\left. + \frac{2\nu}{2-\nu}\partial_\phi(\delta_{1n} + i\,{\rm sign}\,k_z\delta_{3n})\right]\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}{\rm d}k_x\,{\rm d}k_y{\rm e}^{ik_x x}{\rm e}^{ik_y y}\frac{1}{(1+k^2)\sqrt{1-ik_x}}. \tag{58}$$

Note that $\partial_3 = i\,{\rm sign}\,k_z$.

Appendix C. The generating integral

We need to evaluate the generating integral in Eq. (58). Since this integral is not a standard integral, a brief outline of its evaluation will be given in the following. First, integration with respect to k_y (which can be easily obtained by integration along a closed contour in the upper or lower complex plane, depending on the sign of y) yields

$$I = \frac{1}{4\pi}\int_{-\infty}^{\infty}{\rm d}k_x{\rm e}^{ik_x x}{\rm e}^{-|y|\sqrt{1+k_x^2}}\frac{1}{\sqrt{1+k_x^2}\sqrt{1-ik_x}}. \tag{59}$$

Substitution $k_x = \sinh u$ and translation of u by $i\pi/2$ gives

$$I=\frac{1}{4\sqrt{2}\pi}\int_{-\infty}^{\infty}\mathrm{d}u\mathrm{e}^{ir\sinh\left(u+i\frac{\pi}{2}\right)}\cosh\left(\frac{u}{2}-i\frac{|\phi|}{2}\right)$$

$$=\frac{1}{4\sqrt{2}\pi}\int_{-\infty}^{\infty}\mathrm{d}u\mathrm{e}^{-r}\mathrm{e}^{-2r\sinh^2\frac{u}{2}}\frac{\cosh\frac{u}{2}\cos\frac{\phi}{2}+i\sinh\frac{u}{2}\sin\frac{\phi}{2}}{\cos^2\frac{\phi}{2}+\sinh^2\frac{u}{2}}, \tag{60}$$

where $x+iy=re^{i\phi}$. Back substitution $t = \sinh u/2$ gives a standard integral which can be found in Abramovitz and Stegun (1972),

$$I=\frac{1}{2\sqrt{2}\pi}\mathrm{e}^{-r}\sqrt{2r}\cos\frac{\phi}{2}\int_0^{\infty}\mathrm{d}t\frac{\mathrm{e}^{-t^2}}{\left(\sqrt{2r}\cos\frac{\phi}{2}\right)^2+t^2}=\frac{1}{2\sqrt{2}}\mathrm{e}^{r\cos\phi}\mathrm{erfc}\left(\sqrt{2r}\cos\frac{\phi}{2}\right), \tag{61}$$

where erfc denotes the complementary error function.

Appendix D. Self-adjointness of the operator of linear elasticity

In this section, we shall prove that the operator E of linear elasticity is self-adjoint. First, we assume that $\underline{g}$ and $\underline{f}$ satisfy the boundary conditions on the crack surface, i.e.,

$$\underline{\underline{\sigma}}[\underline{g}]\cdot\underline{n}=0$$

and

$$\underline{\underline{\sigma}}[\underline{f}]\cdot\underline{n}=0, \tag{62}$$

where $\underline{n}$ is the normal of the crack surface and $\underline{\underline{\sigma}}[\underline{g}]$ is the stress field generated by $\underline{g}$, i.e., $\underline{\underline{\sigma}}[\underline{g}]=\mathbf{C}\!:\nabla\underline{g}$. Second, we require that $\underline{g}$ and $\underline{f}$ decay sufficiently fast (or, alternatively, that $\underline{g}$ and $\underline{f}$ have compact support). These two conditions ensure that the surface integrals in Eqs. (64) and (66) vanish. The divergence theorem yields

$$\langle\underline{f},E\underline{g}\rangle_2=\int_{\Omega}\mathrm{d}^3x\underline{f}\cdot(\nabla\cdot\mathbf{C}\!:\!\nabla\underline{g}), \tag{63}$$

$$\langle\underline{f},E\underline{g}\rangle_2=\int_{\partial\Omega}\mathrm{d}\underline{S}\cdot(\mathbf{C}\!:\!\nabla\underline{g})\cdot\underline{f}-\int_{\Omega}\mathrm{d}^3x(\nabla\underline{f})\!:\!\mathbf{C}\!:\!(\nabla\underline{g}), \tag{64}$$

$$\langle\underline{f},E\underline{g}\rangle_2=-\int_{\Omega}\mathrm{d}^3x(\nabla\underline{g})\!:\!\mathbf{C}\!:\!(\nabla\underline{f}), \tag{65}$$

$$\langle\underline{f},E\underline{g}\rangle_2=\int_{\Omega}\mathrm{d}^3x\underline{g}\cdot(\nabla\cdot\mathbf{C}\!:\!\nabla\underline{f})-\int_{\partial\Omega}\mathrm{d}\underline{S}\cdot(\mathbf{C}\!:\!\nabla\underline{f})\cdot\underline{g} \tag{66}$$

and

$$\langle\underline{f},E\underline{g}\rangle_2=\langle E\underline{f},\underline{g}\rangle_2. \tag{67}$$

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